Acoustic and Elastic Modeling Using Bases for Bandlimited Functions

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SUMMARY

In this paper we describe a novel algorithm for solving acoustic and elastic wave equations. The algorithm utilizes prolate spheroidal wavefunctions for representing bandlimited functions. Key features of the method are high accuracy, elimination of numerical dispersion, and close to optimal sampling rates. Numerical results in two and three dimensions are presented which demonstrate the accuracy of the method and its ability to handle complex media such as the SEG salt model.

INTRODUCTION

In this paper we demonstrate the use of bases of bandlimited functions in algorithms for wave propagation. Using bandlimited functions allows us to achieve a low sampling rate while significantly reducing numerical dispersion and maintaining high accuracy.

Using bases for bandlimited functions is a significant departure from typical approaches in numerical modeling of seismic wave phenomena. The standard notion of the order of approximation found in methods such as finite differences is not appropriate in our construction, since the basis itself is generated for a finite but arbitrary accuracy across the desired bandwidth. Our method does not have the accuracy and numerical dispersion issues of finite differences, the extreme time step restriction of Chebyshev-PS methods, nor the restriction to periodic functions and difficulties with discontinuous functions of Fourier-PS methods.

We provide a brief summary of the necessary mathematical background, followed by accuracy comparisons with finite difference methods, and some results using the SEG-EAGE salt model.

PROLATE SPHEROIDAL WAVE FUNCTIONS

In this section we summarize the necessary mathematical background. For a more complete discussion, see Beylkin and Sandberg (2005).

In physical wave propagation problems, there is always a bound for both the spatial/time extent and the wavenumber/frequency range. However, a function cannot be compactly supported in both space and the Fourier domain. This motivates the introduction of a basis that can efficiently represent functions of the type e^{ibx} for arbitrary real value b, such that $|b| < c$, where c is a fixed parameter, the bandlimit. Prolate spheroidal wave functions (PSWF), introduced in Slepian and Pollak (1961), Landau and Pollak (1961), Landau and Pollak (1962),Slepian (1964), and Slepian (1978), provide an eigensystem of the bandlimiting operator with bandlimit $[-c, c]$ and maximally concentrated in the interval $[-1, 1]$. Even though PSWF's have been known for some time, until recently there were significant numerical difficulties in using them for practical computations.

For any desired accuracy ε and bandlimit c, we use the PSWF's to construct a set of *nodes* on the interval $[-1, 1]$, such that bandlimited functions may be approximated with accuracy ε on the interval by sampling the function at the nodes. For any accuracy desired, the number of nodes approaches the theoretical Nyquist limit as the number of nodes increases.

We also construct quadrature weights and differentiation matrices so that we may integrate and differentiate bandlimited functions, also with accuracy ε across the entire bandwidth. This is significantly different from existing methods such as finite differences or Chebyshev pseudospectral methods, which have higher accuracy near the lowest wavenumbers at the cost of substantially reduced accuracy as the wavenumber increases. Furthermore, unlike finite differences, our method does not require any special constraints or loss of accuracy near boundaries with pressure-free or pressure-release conditions.

ACCURACY OF THE METHOD

In this section we compare the accuracy of the PSWF-based method with a 4th order in time, 10th order in space finite difference method from Dablain (1986). We solve the constant velocity acoustic equation on a square domain in two dimensions. We use periodic boundary conditions in order to eliminate any possible boundary errors in the finite difference method (the PSWF method suffers no such errors). The initial wave function consists of a single wave number in both directions. We sample at 2.4 points per wavelength in the PSWF method, and a variety of sampling rates ranging from 4 to 12 points per wavelength. As can be see in Figure 1, even after 100 wavelengths of propagation in time, the PSWF method maintains more than 3 digits (about 70 dB) of accuracy, whereas the finite difference method is not able to achieve even 2 digits of accuracy using 12 points per wavelength. In other words, at 5 times the sampling rate, the finite difference method has an error which is 40 dB larger.

We also consider point source in two dimensions on a square domain. The source waveform is a Ricker wavelet, constructed so that the sampling rate at peak frequency is 5.1 points per wavelength for the PSWF method, and either 5.1 or 10.2 points per wavelength for the finite difference method.

Figure 2 shows a profile of the wavefront after propagating approximately 95 wavelengths in space, using 5.1 points per wavelength for the PSWF method, and the same sampling rate for the finite difference method. The PSWF solution shows no visible dispersion, whereas the finite difference solution shows dispersion at a very high level.

Increasing the sampling rate to 10.2 points per wavelength for the finite difference solution improves the dispersion somewhat, but as can be seen in Figure 3, a substantial amount of dispersion still exists.

We note that some methods have been proposed (see e.g. Fei and Larner (1995)) for dealing with numerical dispersion in finite difference methods, however such methods are completely ad-hoc and offer no estimates, either theoretical or emperical, as to the accuracy.

SEG SALT MODEL EXAMPLES

We consider part of the SEG-EAGE salt model, a vertical slice of which is shown in Figure 4. A Ricker point source with peak frequency 20 Hz is placed just above the salt body, approximately centered laterally. Using the velocity at the source, this corresponds to a sampling rate of roughly 4.9 points per wavelength at peak frequency (though the sampling rate may change as the wavefront moves into regions of higher or lower velocity than at the source).

We propagate using the PSWF method and the 4/10 finite difference method for about 10 wavelengths; Figures 5 and 6 show profiles of each solution at that point in time (note that although we display only two dimensional images, all computations were done on a three-dimensional grid). As expected, the finite difference solution shows levels of numerical dispersion. In complex regions such as below the salt body, the effects of dispersion are especially pernicious, since one cannot easily identify which part of the solution consists of dispersion and which part is the“true” wavefield.

Increasing the sampling rate to 9.8 points per wavelength in the finite difference solution, as shown in Figure 7, improves the appearance of the wave field significantly. However, without any accuracy estimates we cannot be sure of the amplitudes, or of any dispersive effects that simply are not readily visible in a low-resolution picture such as must be presented in this format. The accuracy study shown in Figure 1 shows clearly that even at double the sampling rate, we expect much lower accuracy from the finite difference method.

CONCLUSION AND SUMMARY

Prolate spheroidal wave functions offer many significant advantages over existing methods when for numerical solution of wave equations. In particular, using such functions allows us to:

- achieve a sampling rate that is near the Nyquist rate
- achieve substantially higher accuracy than with methods such as finite difference methods
- eliminate numerical dispersion
- propagate over relatively long distances

FIGURES

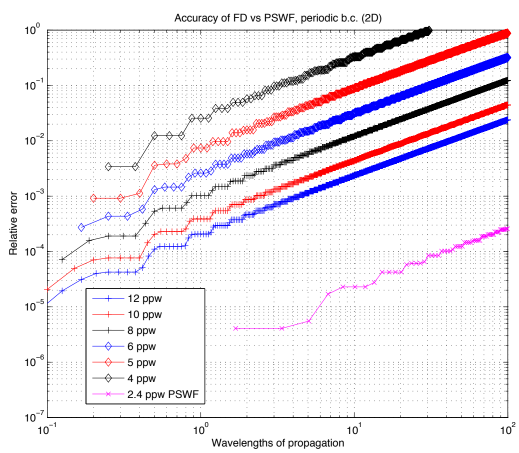

Figure 1: Two dimensional, constant velocity example. The PSWF method achieves substantially higher accuracy at a lower sampling rate than the 4/10 finite difference method.

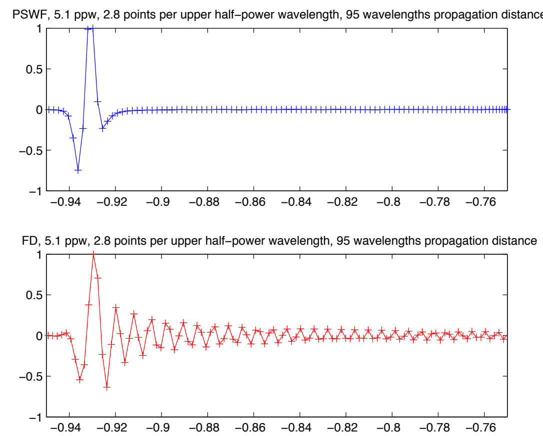

Figure 2: Two dimensional, constant velocity example with Ricker point source. Shown is a profile of the wavefront. The PSWF method exhibits no visible dispersion; the 4/10 finite difference solution, at the same sampling rate of 5.1 points per wavelength at peak frequency, shows unacceptable dispersive effects.

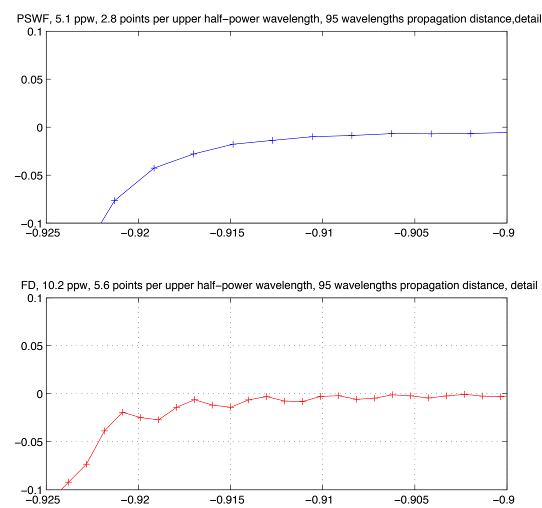

Figure 3: Two dimensional, constant velocity example with Ricker point source. Shown is a profile of the wavefront. The PSWF method exhibits no visible dispersion; the 4/10 finite difference solution, double the sampling rate of the PSWF solution,still shows substantial dispersive effects.

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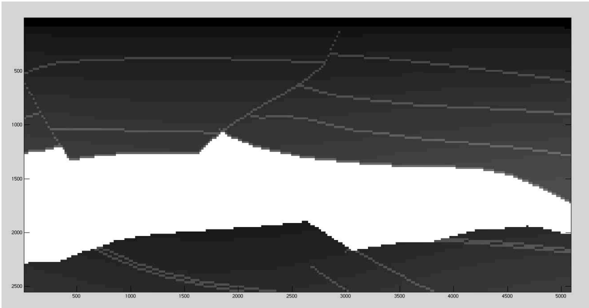

Figure 4: SEG salt model velocity profile.

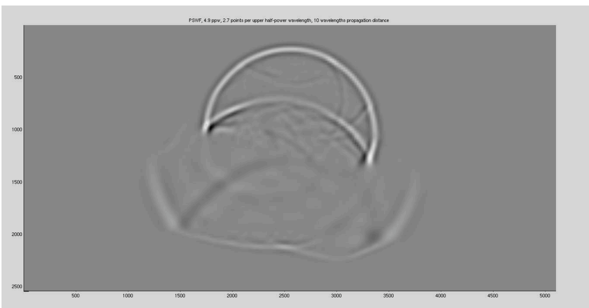

Figure 5: PSWF solution for SEG salt model at 4.9 points per wavelength peak frequency sampling rate.

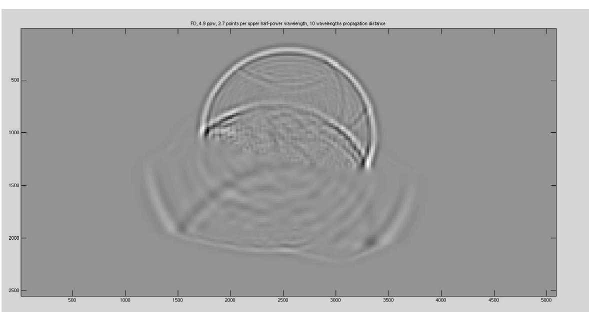

Figure 6: Finite difference solution for SEG salt model at 4.9 points per wavelength peak frequency sampling rate.

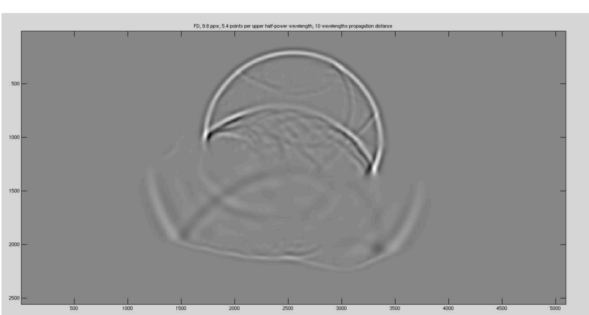

Figure 7: Finite difference solution for SEG salt model at 9.8 points per wavelength peak frequency sampling rate.

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